

Roll No.

D-983**M. A./M. Sc. (Fourth Semester) (Main/ATKT)****EXAMINATION, May-June, 2020**

MATHEMATICS

Paper First

(Functional Analysis—II)*Time : Three Hours]**[Maximum Marks : 80***Note :** Attempt all Sections as directed.**Section—A**

1 each

(Objective/Multiple Choice Questions)**Note :** Attempt all questions.

Choose the correct answer :

1. Let T be a closed linear map of Banach space X into Banach space Y , then :
- T is closed
 - T is open
 - T is continuous
 - All of the above

2. Let X be an arbitrary normed linear space the mapping $f : X \rightarrow X^{**}$ is isometric isomorphics from X into X^{**} if :
- It is linear
 - It is bounded
 - It preserves distance
 - All of the above
3. Let X and Y be normed linear space and $D \subset X$, then linear transformation $T : D \rightarrow Y$ is closed if and only if its graph G_T is :
- open
 - closed
 - bounded
 - None of these
4. Let $\{T_n\}$ be a sequence of continuous linear operator of Banach space X into Banach space Y such that $\lim_{n \rightarrow \infty} T_n x = Tx$ exists for every $x \in X$, then T is continuous linear operator and :
- $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$
 - $\|T\| \leq \limsup_{n \rightarrow \infty} \|T_n\|$
 - $\|T\| \geq \liminf_{n \rightarrow \infty} \|T_n\|$
 - $\|T\| \geq \limsup_{n \rightarrow \infty} \|T_n\|$

P. T. O.

5. Let X be a normed space over field K and S be a linear subspace of X . Suppose that $Z \in X$ and $\text{dist}(Z, S) = d > 0$, then there exist $g \in X^*$ such that :
- $g(s) = \{0\}, g(z) = d, \|g\| = 1$
 - $g(s) \neq \{0\}, g(z) = d, \|g\| = 1$
 - $g(s) = \{0\}, g(z) = d, \|g\| \neq 1$
 - $g(s) \neq \{0\}, g(z) = d, \|g\| \neq 1$
6. Let X be a normed space, then the set of all bounded linear functional on X constitutes a normed space with the norm defined by :
- $\|f\| = \sup \left\{ \frac{|f(x)|}{\|x\|} : x \in X; x \neq 0 \right\}$
 - $\|f\| = \sup \{ |f(x)| : x \in X; \|x\| = 1 \}$
 - Both (a) and (b)
 - None of these
7. Let X and Y be normed space over the field K and $T : X \rightarrow Y$ be a bounded linear operator, then :
- The adjoint T^* is bounded linear from Y^* to X^*
 - $\|T^*\| = \|T\|$
 - The mapping T of T^* is an isometric isomorphism of $B(X, Y)$ into $B(Y^*, X^*)$
 - All of the above

P. T. O.

8. Let Y be a linear subspace of normed linear space X and let f be a functional defined on Y , then f can be extended to functional F defined on the whole space X such that :
- $\|f\| = \|F\|$
 - $\|f\| \neq \|F\| = 1$
 - $\|f\| \neq \{0\}, \|F\| = 1$
 - None of these
9. If x and y are any two vectors in an inner product space X , then :

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

The above inequality is known as :

- Parallelogram law
 - Cauchy-Schwarz's inequality
 - Polarisation identity
 - Bessel's inequality
10. A normed space is an inner product space if and only if the norm of the normed space satisfy the equation.
- $\|x + y\| \leq \|x\| + \|y\|$
 - $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$
 - $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$
 - $\langle x, y \rangle = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2]$,
where $K = \mathbb{R}$

11. If A is a subset of an inner product space X , then which statement is incorrect ?

- (a) $A \subseteq A^{\perp\perp}$
- (b) $(A^{\perp})^{\perp} = A^{\perp\perp}$
- (c) $A = A^{\perp\perp}$
- (d) None of these

12. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H , then :

$$\|x\|^2 = \sum | \langle x, e_i \rangle |^2$$

This equality is known as :

- (a) Parseval's identity
- (b) Apollonius identity
- (c) Holder's inequality
- (d) None of these

13. Let X be an inner product space. Which is incorrect statement ?

- (a) If $x \perp y \Leftrightarrow y \perp x, \forall x, y \in X$
- (b) $x \perp 0, \forall x \in X$
- (c) 0 is the only vector in X which is orthogonal to itself.
- (d) $A \cap A^{\perp}$ is neither $\{0\}$ nor ϕ

i. e. $A \cap A^{\perp} \neq \{0\}$ or ϕ

P. T. O.

14. A subspace M of a Hilbert space H is closed in H if :

- (a) $M = M^{\perp\perp}$
- (b) $H = M \oplus M^{\perp}$
- (c) Both (a) and (b)
- (d) None of these

15. A Banach space X is said to be reflexive if it is isometrically isomorphic to :

- (a) X^*
- (b) X^{**}
- (c) X^{***}
- (d) All of the above

16. Let H_1 and H_2 be Hilbert space and T and S are element of $B(H_1, H_2)$ and $\alpha \in K$, then which statement is incorrect ?

- (a) $(T + S)^* = T^* + S^*$
- (b) $(TS)^* = S^* T^*$
- (c) $T^{**} = T$
- (d) $(\alpha T)^* = \alpha T^*$

17. An operator T is called unitary if :

- (a) $T = T^*$
- (b) $T^*T = TT^*$
- (c) $T^*T = TT^* = I$
- (d) $\langle T_x, x \rangle \geq 0, \forall x \in H$

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18. If T is a positive operator on a Hilbert space H , then :

- (a) $I + T$ is non-singular
- (b) $I - T$ is non-singular
- (c) $I + T$ is singular
- (d) None of these

19. Let $P \in B(H)$ be a projection operator, then :

- (a) $R(P)$ and $N(P)$ are closed subspace of H
- (b) $I - P$ is a projection
- (c) $R(P) = N(I - P)$
- (d) All of the above

20. Which statement is incorrect ?

- (a) Every positive operator is self-adjoint.
- (b) Every self-adjoint operator is normal.
- (c) Every normal operator is unitary.
- (d) Every unitary operator is normal.

Section—B

2 each

(Very Short Answer Type Questions)

Note : Attempt all questions. Answer in 2 to 3 sentences.

1. Give an example of a closed operator which is not bounded.
2. Define dual space and normed linear space.
3. Define weak sequential compactness.
4. Define inner product space.

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5. If A and B are subset of inner product space X such that

$A \subset B$ then show that $A^\perp \supset B^\perp$.

6. Write the statement of Riesz representation theorem.

7. Prove that every positive operator is self-adjoint.

8. Define normal operator on Hilbert space.

Section—C

3 each

(Short Answer Type Questions)

Note : Attempt all questions. Answer in 75 words.

1. Let X be a Banach space over the field K . If :

$$\{T_n\} \in B(X, Y)$$

be a sequence such that :

$$\lim_{n \rightarrow \infty} T_n x = Tx$$

where $x \in X$ exists, then prove that :

$$T \in B(X, Y).$$

2. Show that a Banach space is reflexive if and only if its dual space is reflexive.

3. Show that in an inner product space X $x \perp y$ if and only if

$$\|x + \alpha y\| \geq \|x\| \quad \forall \text{ scalar } \alpha \quad \forall x, y \in X.$$

4. Show that the linear space l^p , where $1 \leq p < \infty$ and $p \neq 2$ where norm is defined by :

$$\|x\|_p = \left(\sum |\xi_i|^p \right)^{\frac{1}{p}}$$

$$x = \langle \xi_i \rangle \in l^p$$

is not an inner product space and hence it is not a Hilbert space.

5. Let H_1 and H_2 be Hilbert space and $T \in B(H_1, H_2)$, then prove that :

$$\|T^* T\| = \|T\|^2 = \|T T^*\|$$

6. Show that the mapping :

$$\psi : H \rightarrow H^*$$

defined by $\psi(y) = f_y$, where $f_y(x) = \langle x, y \rangle$ is one-one onto but not linear and an isometry.

7. Show that the product of two bounded self-adjoint operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute.

P. T. O.

8. Let H be complete Hilbert space and $T \in B(H)$, then the following statements are equivalent :

(a) T is normal

(b) T^* is normal

(c) $\|T^* x\| = \|T x\| \quad \forall x \in H$

Section—D

5 each

(Long Answer Type Questions)

Note : Attempt all questions. Answer in 150 words.

1. State and prove closed graph theorem.

Or

State and prove uniform boundedness principle.

2. Let M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M , then there exist a functional f_0 in N^* such that :

$$f_0(M) \neq 0$$

and

$$f_0(x_0) \neq 0.$$

Or

State and prove closed rang theorem for Banach space.

3. Prove that every Hilbert space is reflexive.

Or

Show that a closed convex subset C of Hilbert space H contains a unique vector of smallest norm.

4. Let y be a fixed vector in Hilbert space H and let f_y be scalar defined by :

$$f_y(x) = \langle x, y \rangle \forall x \in H$$

then show that :

$$f_y \in B(H, K)$$

i.e. $f_y \in H^*$.

Further show that :

$$\|y\| = \|f_y\|$$

Or

A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometry and surjective.