GS2023 - Mathematics Question Paper

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm of x, i.e., the distance from x to 0.
- All rings are associative, with a multiplicative identity.
- For any ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space. $\mathrm{M}_n(\mathbb{R})$ is given the topology such that any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ is a homeomorphism. Subsets of $\mathrm{M}_n(\mathbb{R})$ are given the subspace topology.
- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a finite group, and let $S \subset G$. We say that S generates G if no proper subgroup of G contains S.
- If $f : X \to Y$ is a map of sets, and $X_1 \subset X$, then $f|_{X_1}$ denotes the restriction of f to X_1.

PART A

Answer the following multiple choice questions.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = (3x^2+1)/(x^2+3)$. Let $f^{\circ 1} = f$, and let $f^{\circ n} = f^{\circ(n-1)} \circ f$ for all integers $n \geq 2$. Which of the following statements is correct?

   (a) $\lim_{n\to\infty} f^{\circ n}(1/2) = 1$, and $\lim_{n\to\infty} f^{\circ n}(2) = 1$.

   (b) $\lim_{n\to\infty} f^{\circ n}(1/2) = 1$, but $\lim_{n\to\infty} f^{\circ n}(2)$ does not exist.

   (c) $\lim_{n\to\infty} f^{\circ n}(1/2)$ does not exist, but $\lim_{n\to\infty} f^{\circ n}(2) = 1$.

   (d) Neither $\lim_{n\to\infty} f^{\circ n}(1/2)$ nor $\lim_{n\to\infty} f^{\circ n}(2)$ exists.

2. Consider the following properties of a sequence $\{a_n\}_n$ of real numbers.

   (I) $\lim_{n\to\infty} a_n = 0$.

   (II) There exists a sequence $\{i_n\}_n$ of positive integers such that $\sum_{n=1}^{\infty} a_{i_n}$ converges.

   Which of the following statements is correct?

   (a) (I) implies (II), and (II) implies (I).

   (b) (I) implies (II), but (II) does not imply (I).

   (c) (I) does not imply (II), but (II) implies (I).

   (d) (I) does not imply (II), and (II) does not imply (I).

3. Consider sequences $\{x_n\}_n$ of real numbers such that

   $$\lim_{n\to\infty} (x_{2n-1} + x_{2n}) = 2 \qquad \text{and} \qquad \lim_{n\to\infty} (x_{2n} + x_{2n+1}) = 3.$$

   Which of the following statements is correct?

   (a) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = 1$.

   (b) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = -1$.

   (c) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = 3/2$.

   (d) There exists such a sequence $\{x_n\}_n$, for which $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}}$ does not exist.

4. Consider the function $f : (0,\infty) \to (0,\infty)$ given by $f(x) = xe^x$. Let $L : (0,\infty) \to (0,\infty)$ be its inverse function. Which of the following statements is correct?

   (a) $\lim_{x\to\infty} \frac{L(x)}{\log x} = 1$.

   (b) $\lim_{x\to\infty} \frac{L(x)}{(\log x)^2} = 1$.

   (c) $\lim_{x\to\infty} \frac{L(x)}{\sqrt{\log x}} = 1$.

   (d) None of the remaining three options is correct.

5. Let $\{b_n\}_n$ be a monotonically increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$. Which of the following statements is true about

   $$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \frac{b_k}{k^2}?$$

(a) The limit exists for all such sequences, and its value is always $+\infty$.

(b) The limit exists for all such sequences, and its value is always 0.

(c) The limit exists for all such sequences, and its value is always 1.

(d) None of the remaining three options is correct.

6. For every positive integer n, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \frac{\sin(n^2 x)+\cos(e^n x)}{1+n^2x^2}$. Then

$$\lim_{n\to\infty} \int_0^{1-\sin(1/n)} f_n(x)\,dx$$

equals

(a) 1.

(b) 0.

(c) ∞.

(d) 1/2.

7. Consider the functions $f_1, f_2 : (0,\infty) \to \mathbb{R}$ defined by

$$f_1(x) = \sqrt{x}, \qquad \text{and} \qquad f_2(x) = \sqrt{x}\sin x.$$

Which of the following statements is correct?

(a) f_1 and f_2 are uniformly continuous.

(b) f_1 is uniformly continuous, but f_2 is not.

(c) f_2 is uniformly continuous, but f_1 is not.

(d) Neither f_1 nor f_2 is uniformly continuous.

8. Let $x_1 \in \mathbb{R}^2 \setminus \{0\}$ be fixed, and inductively define $x_{n+1} = Ax_n$ for $n \geq 1$, where A is the 2×2 real matrix given by

$$A := \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Which of the following statements is correct?

(a) $\{x_n\}_n$ is a convergent sequence.

(b) $\{x_n\}_n$ is not a convergent sequence, but it has a convergent subsequence.

(c) $\lim_{n\to\infty} \|x_n\| = 0$.

(d) None of the remaining three options is correct.

9. Let $T : \mathrm{M}_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear map defined by $T(A) = A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Then the dimension of the kernel of T equals

(a) 2.

(b) 8.

(c) 1.

(d) None of the remaining three options.

10. Let $V = \{f(x) \in \mathbb{R}[x] \mid f(0) = 0\}$, viewed as a real vector space. Consider the following assertions:

(I) V contains three linearly independent polynomials of degree 2.

(II) V contains two linearly independent polynomials of degree 3.

Which of the following statements is correct?

(a) Both (I) and (II) are true.

(b) (I) is true, but (II) is false.

(c) (I) is false, but (II) is true.

(d) Neither (I) nor (II) is true.

11. Let $C([-1,1],\mathbb{R})$ denote the real vector space of continuous functions from $[-1,1]$ to $\mathbb{R}$, and consider the subspace

$$V = \{f \in C([-1,1],\mathbb{R}) \mid f(-x) = f(x) \text{ for all } x \in [-1,1]\}.$$

Define an inner product on $C([-1,1],\mathbb{R})$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

What is the orthogonal complement of V in $C([-1,1],\mathbb{R})$?

(a) $\{f \in C([-1,1],\mathbb{R}) \mid f(-x) = -f(x) \text{ for all } x \in [-1,1]\}$.

(b) $\{f \in C([-1,1],\mathbb{R}) \mid f(0) = 0\}$.

(c) V does not have an orthogonal complement in $C([-1,1],\mathbb{R})$.

(d) None of the remaining three options.

12. Consider pairs (X,d), where X is a set with 100 elements, and $d : X \times X \to \mathbb{R}$ is a function such that $d(x,y) = d(y,x) > 0$ if $x,y \in X$ are distinct, and $d(x,x) = 0$ for all $x \in X$. For $n < 100$, let A_n be the statement:

For every such pair (X,d), there exists a subset X_1 of X, with n elements, such that $(X_1, d|_{X_1 \times X_1})$ is a metric space.

Which of the following statements is correct?

(a) A_2 is true, but A_3 is not true.

(b) A_3 is true, but A_4 is not true.

(c) A_n is true for all $n \leq 10$, but not for all $n \leq 25$.

(d) A_n is true for all $n \leq 25$.

13. Let $\{x_n\}_n$ be a sequence in a metric space (X,d). Let $f : X \to \mathbb{R}$ be defined by

$$f(x) = \inf\{d(x,x_n) \mid n \in \mathbb{N}\}.$$

Which of the following statements is correct?

(a) f is uniformly continuous on X.

(b) f is continuous on X, but not necessarily uniformly continuous.

(c) f is continuous on X if and only if X is compact.

(d) None of the remaining three options is correct.

14. The number of finite groups, up to isomorphism, with exactly two conjugacy classes, equals

(a) 1.

(b) 2.
(c) Greater than 2, but finite.
(d) Infinite.

15. Consider the following assertions about a commutative ring R with identity and elements $a, b \in R$:

 (I) There exist $p, q \in R$ such that $ap + bq = 1$.
 (II) There exist $p, q \in R$ such that $a^2p + b^2q = 1$.

 Then:

 (a) (I) implies (II), and (II) implies (I).
 (b) (I) implies (II), but (II) does not imply (I).
 (c) (I) does not imply (II), but (II) implies (I).
 (d) (I) does not imply (II), and (II) does not imply (I).

16. The number of elements of finite order in the group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

 is

 (a) 1.
 (b) Finite, but not 1.
 (c) Countably infinite.
 (d) Uncountably infinite.

17. The value of

$$\max \left(\bigcup_{\substack{k \in \mathbb{N} \\ k \geq 1}} \{x_1 x_2 \dots x_k \mid x_1, \dots, x_k \in \mathbb{N}, \text{ and } x_1 + \dots + x_k = 100\} \right)$$

 equals

 (a) 4×3^{32}.
 (b) 2^{50}.
 (c) $2^{26} \times 3^{16}$.
 (d) None of the remaining three options.

18. Choose the option that completes the sentence correctly: There exists a 10×10 real symmetric matrix A, all of whose entries are nonnegative and all of whose diagonal entries are positive, such that A^{10} has

 (a) exactly 67 positive entries.
 (b) exactly 68 positive entries.
 (c) exactly 69 positive entries.
 (d) exactly 70 positive entries.

19. The number of (nondegenerate Euclidean) triangles with sides of integer length and perimeter 8, up to congruence, is

(a) 1.
(b) 2.
(c) 3.
(d) 4.

20. Let

$$A = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid \text{ the roots } r_1, r_2, r_3 \text{ of the polynomial } p(x) = x^3 - 2x^2 + \alpha x - \beta \text{ satisfy } r_1^3 + r_2^3 + r_3^3 = 0\}.$$

Which of the following statements is correct?

(a) A is infinite.
(b) A is empty.
(c) A is singleton.
(d) A is finite, but neither empty nor singleton.

PART B

Answer whether the following statements are True or False.

1. Let α be a positive real number, and let $f : (0,1) \to \mathbb{R}$ be a function such that $|f(x)-f(y)| \leq |x-y|^\alpha$ for all $x, y \in (0,1)$. Then f can be extended to a continuous function $[0,1] \to \mathbb{R}$. *True*

2. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f^2 + g^2$ is uniformly continuous. Then at least one of the two functions f and g is uniformly continuous. *False*

3. Let $\{f_n\}_n$ be a sequence of (not necessarily continuous) functions from $[0,1]$ to $\mathbb{R}$. Let $f : [0,1] \to \mathbb{R}$ be such that for any $x \in [0,1]$ and any sequence $\{x_n\}_n$ consisting of elements from $[0,1]$, if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$. Then f is continuous. *True*

4. Let $A, B \in \mathrm{M}_2(\mathbb{Z}/2\mathbb{Z})$ be such that $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\mathrm{tr}(A^2) = \mathrm{tr}(B^2)$. Then A and B have the same eigenvalues. *False*

5. Let v_1, v_2, w_1, w_2 be nonzero vectors in $\mathbb{R}^2$. Then there exists a 2×2 real matrix A such that $Av_1 = v_2$ and $Aw_1 = w_2$. *False*

6. Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ be such that $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$. Assume that $\lim_{m\to\infty} A^m$ exists, and denote it by $B = (b_{ij})$. Then, for all $1 \leq i, j \leq n$, we have $b_{ij} \in \{0, 1\}$. *False*

7. Given any monic polynomial $f(x) \in \mathbb{R}[x]$ of degree n, there exists a matrix $A \in \mathrm{M}_n(\mathbb{R})$ such that its characteristic polynomial equals f. *True*

8. If $A \in \mathrm{M}_4(\mathbb{Q})$ is such that its characteristic polynomial equals x^4+1, then A is diagonalizable in $\mathrm{M}_4(\mathbb{C})$. *True*

9. If $A \in \mathrm{M}_n(\mathbb{R})$ is such that $AB = BA$ for all invertible matrices $B \in \mathrm{M}_n(\mathbb{R})$, then $A = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{R}$. *True*

10. There exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(2x) = 3f(x)$ for all $x \in \mathbb{R}$. *True*

11. There exists a continuous bijection from $[0,1] \times [0,1]$ to $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, which is not a homeomorphism. ***False***

12. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a nonzero polynomial ($n \geq 1$), and let *True*

$$X = \{z \in \mathbb{C}^n \mid f(z) = 0\}.$$

Then $\mathbb{C}^n \setminus X$ is path connected.

13. A connected metric space with at least two points is uncountable. *True*

14. If A and B are disjoint subsets of a metric space (X, d), then ***False***

$$\inf\{d(x, y) \mid x \in A, y \in B\} \neq 0.$$

15. A countably infinite complete metric space has infinitely many isolated points (an element x of a metric space X is said to be an isolated point if $\{x\}$ is an open subset of X). *True*

16. Suppose G and H are two countably infinite abelian groups such that every nontrivial element of $G \times H$ has order 7. Then G is isomorphic to H. *True*

17. There exists a nonabelian group G of order 26 such that every proper subgroup of G is abelian. *True*

18. Let G be a group generated by two elements x and y, each of order 2. Then G is finite. ***False***

19. $\mathbb{R}[x]/(x^4 + x^2 + 2023)$ is an integral domain. ***False***

20. Every finite group is isomorphic to a subgroup of a finite group generated by two elements. *True*